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Heterogeneity Characterization of Recursive Line Networks

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ABSTRACT
Over the past few years, the study of complex networks as an interdisciplinary subject has yielded numerous insights. Communication links within these networks have been found to play a crucial role in shaping the implementation of dynamic processes. Recursive graphs are a class of complex networks whose internal structure is governed by recurrent relations. Among these, line graphs are especially important because they represent the communication links within the network as nodes. Studying the heterogeneity, or irregularity, of different graph models is a fundamental research issue in complex and social network analysis. In this article, we investigate the mapping between graph robustness and heterogeneity metrics and their equivalent metrics in line graphs. Specifically, we analyze the distribution of eigenvalues and important indices of heterogeneity in recursive and line graphs. We also examine the changes in heterogeneity of recursive line graphs with the introduction of a set of important heterogeneity indices. Our approach is broadly applicable to a wide range of indicators and complex networks beyond those discussed in this study.

KEYWORDS
Complex and Social networks, Recursive networks, Line graph, Heterogeneity measures

1. INTRODUCTION

Complex networks are a fundamental concept in various complex systems, including social and ecological, biological, and technological systems. Interdisciplinary research in this field is crucial in creating and adapting new models that explore and study these networks using scientific and engineering concepts. Scholars have played a critical role in recognizing and analyzing the dynamic and topological properties of complex networks. The growth of network approaches is attributed to the emergence and expansion of dynamic systems and various inter-system connections that can be analyzed using statistical and mathematical techniques rooted in graph theory. The applications of graph theory are diverse and range from urban planning and traffic control to epidemiology, financial planning, internet search engines, analysis of complex environmental and molecular biological systems, and psychometrics. While classical graph theory mainly deals with the analysis of random graphs, modern network science focuses on real-world systems and has shown that real networks are not random in most cases. These networks have basic infrastructure processes that ensure their survival, growth, and structure. In graph theory, a complex network refers to a network with special structural features that occur only in graphs based on natural phenomena. These features cannot be observed in random graphs. Complex networks encompass a wide range of networks, including social networks that examine communication networks between humans. Recursive networks are considered a category of synthetic complex networks where the growth factor is done recursively.

Recursive networks, such as Pascal's triangle and the Fibonacci sequence, are commonly used in mathematical sciences due to their simple design and recursive capability. These networks can easily accommodate new nodes without the need for reconfiguration of the overall structure, making them highly scalable. One of the key features of recursive networks is the presence of multiple paths between each pair of nodes, which increases reliability, while some of these paths are optimized for shorter distances to minimize communication delays. Moreover, recursive networks exhibit a strong correlation due to the recursive relationship that governs their structure. It is worth noting that there are different recursive algorithms that can be employed to construct these networks, and various mathematical relationships have been proposed and proven by researchers to enhance the calculation of node labels in recursive networks.

The characterization of similarities and dissimilarities and the exploration of homomorphism among graphs is a crucial issue in the fields of science and engineering, particularly in the theoretical realm of complex and social networks. As graphs contain a vast amount of information and are widely used in social, medical, biological sciences, and other fields, it is essential

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to measure the degree of heterogeneity in these graphs and identify efficient metrics for classifying and characterizing similarities between networks. Such metrics can prove useful in many important and practical cases.

In complex networks, line graphs are a significant area that has received less attention so far. Line graphs are formed by transforming the communication links in complex networks into corresponding nodes. It is worth noting that links in real-world networks are susceptible to damage, such as breakdowns in transmission lines in power grid networks, congestion in transportation network routes, and other similar issues. Line graphs can be used to understand the impact of failures caused by links in infrastructure networks.

In this article, we aim to explore the mapping of adjacency matrices of complex recursive networks to line graphs and analyze the heterogeneity of some important complex line graphs. Throughout the article, the terms "graph" and "network" are used interchangeably. We also introduce a set of important criteria and indicators to analyze heterogeneity. The article is organized into five sections. Section 1 provides an introduction to the topic and outlines the main objectives. Section 2 provides a literature review on heterogeneity, complex recursive networks, and line graphs. Section 3 discusses previous studies and provides an overview of the topic. In Section 4, we analyze the numerical results obtained through simulation experiments. Finally, Section 5 presents the findings and conclusions of the article, along with suggestions for future research in this area.

2. PRELEMINARIES

In this section, a brief review of the related studies and its preparations will be drawn.

2.1 Heterogeneity in Graphs and Complex Networks

According to the definition [1], a graph G is called as homogeneous (regular) if and only of the degree of all its vertices is equal. Conversely, if the degree of vertices in G varies, it is depicted as heterogeneous (irregular) graph. In this study, we denote the set of heterogeneity indicators by IM. This set encompasses various measures of heterogeneity, and any index that estimates the graph heterogeneity may be regarded as a member of the IM set.

The IM set possesses several important features. Firstly, if a heterogeneity index X belongs to IM, then any positive constant multiple of X, denoted by CX, also belongs to IM. This scale-free property of the set enables us to produce new heterogeneity indices by multiplying any existing index with a positive constant. Further, the IM set is closed under addition and multiplication operations, meaning that combining two heterogeneity indices X_1 and X_2 yields another heterogeneity index which belongs to the IM set. This is a crucial feature as it allows us to create composite indices by blending different measures of heterogeneity. By doing so, we can gain a deeper understanding of the degree of network heterogeneity and develop more accurate heterogeneity indices. Consequently, the concept of the IM set can provide a valuable framework for comprehending and quantifying the heterogeneity of graphs and complex networks.

2.2 Complex Recursive Networks

Complex networks have been the subject of study across many scientific disciplines, as they provide valuable models for understanding natural systems. These networks consist of nodes that are connected by links, forming a complex web of interdependent relationships [2]. Recursive networks are a special type of complex network that is characterized by a topology derived from a recursive relationship. These networks are neither completely regular nor completely random, and are particularly useful in the fields of science and engineering. In the following sections, we will describe five important recursive networks and their applications.

2.2.1 Stirling Network

The Stirling number [3] of the first kind, denoted by s(n, k), is a mathematical concept used in various combinatorial problems and analysis. It represents the number of ways to partition a set of n elements into k non-empty cycles, such that each element belongs to exactly one cycle. The value of s(n, k) can be calculated using the following equation:

$$c(n,k) = \begin{bmatrix} n \\ k \end{bmatrix} = |s(n,k)| \tag{1}$$

A family of labeled and undirected graphs can be derived from the table of Stirling numbers of the first kind. These graphs have been extensively studied for their properties and can be used as a basis for connecting various types of networks. The Stirling network has found applications in various fields, such as providing a deadlock-free and congestion-avoiding routing algorithm. It is also used in the construction of very large-scale integration circuits (VLSI) and complex networks. A modular approach has been proposed for constructing these networks [3].

Stirling networks are characterized by their multiple connections between nodes, which provide them with an important feature: fault-tolerance. If a path in the network fails, there will always be an alternative path for messages to pass through. This ability to control errors in the face of faulty nodes or broken links is an important advantage of Stirling networks. In addition, several classes of parallel algorithms can be optimally implemented using such networks.

An undirected graph with n vertices, corresponding to the adjacency matrix SM_n, is called Pascal graph SG_n of order n; where n represents the number of nodes in the Stirling graph. The set of vertices and edges of this graph is represented by $V=\{v_1, v_2,...,v_n\}$ and $E(SG_n)$ respectively. According to the adjacency matrix, the vertices of the SG_n graph are constructed from V_1 to V_n respectively and in each step, one node is added to the graph. The diameter of the Stirling network with n nodes can be calculated by $\left\lceil \log_2^{n+1} \right\rceil - 1$. Stirling networks can grow recursively.

To generate the network adjacency matrix of Stirling network, first the table of Stirling numbers of the first type must be obtained from the following recursive relation. (The initial conditions are valid for $k>0$.) [3]

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = n\begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix}$$
$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1, \begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ n \end{bmatrix} = 0 \quad (2)$$

In other words, the table of Stirling numbers of the first type, $st_{n,k}$ means that it consists of n distinct elements and contains k cycles. Some properties of Stirling numbers are expressed in the following relation (for n>1) [3].

$$st_{n,n} = 1,\ st_{n,1} = (n-1)!,\ st_{n,n-1} = \binom{n}{2},$$
$$st_{n,k} = st_{n-1,k-1} + (n-1)st_{n-1,k},\ 1 \le k \le n \quad (3)$$

The detailed and supplementary information about the Stirling table of the first type and the different orders of these networks, as well as the specifications related to other correlations, are described in [3]. By using the recursive relation governing this network, to build the graph SG_n, the previous step SG_{n-1} should be used. Considering this feature of the graph, the Stirling graph can be implemented and displayed for any order. Figure 1 depicts the Stirling graph of the sixth order (i.e. n=6).

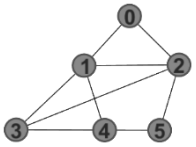

Figure 1. A Stirling network of order sixth (n=6)

2.2.2 Pascal Network

Over the past few decades, researchers have conducted numerous studies on Pascal's graphs and have discovered many new features. The structural aspects of the network were first examined more than twenty years ago, revealing that Pascal's graph possesses unique properties [4]. Blaise Pascal first introduced Pascal's triangle in the mid-17th century, which plays a vital role in constructing Pascal's matrix. Different models of this graph have been used in network integration and topology, including computer networks where its properties are used to assess network stability. The distinctive structural characteristics of Pascal's graph distinguish it from other types of graphs [4].

Pascal graph (PG), which is the same as the Pascal matrix (PM), can be constructed from the Pascal's triangle. The set of vertices and edges of this graph is represented by V={v_1, v_2,...,v_n} and E(PG_n) respectively. According to the adjacency matrix of the graph, the vertices of PG_n graph are constructed from V_1 to V_n, respectively. In each step, one node is added to the graph [5]. Some features of Pascal graphs are as follows:

- Pascal graphs are simple and recursive.
- Pascal's graph $PG(i)$ is planar for 1<i <7; but it is un-planar for high degrees.
- Node V_1 is adjacent to all nodes of Pascal's graph; this means that its first vertex (V_1) is connected to all other adjacent vertices.
- In Pascal graph, node V_i is adjacent to node V_{i+1} for $i \ge 1$
- The vertices of Pascal's graph have at least 2 connections from degree greater than or equal to 3
- In Pascal's graph, none of the even-numbered vertices are adjacent to each other
- $PG_{i,j}$ refers to the j^{th} element of the i^{th} row of the Pascal's triangle; where rows and elements begin with zero. This network can be obtained from the following recursive relation [5].
-

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k},\ 0 < k < n$$
$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad (4)$$

Pascal network has several properties and features that have been described with details in [6]. In addition, in [6], constructing the adjacency matrix and different orders of the network has been explained. Figure 2 depicts a Pascal network with the order of eight (n=8).

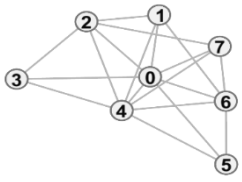

Figure 2. A Pascal network of order eight (n=8)

2.2.3 Binomial Network

The investigation of Pascal's triangle with entries reduced to module 2 has been a subject of numerous researches. Studies have been conducted on the parity of binomial coefficients, the geometrical structure of the binomial graph, and its similarity with the Sierpinski triangle [7]. The adjacency matrix of the binomial graph is constructed from Pascal's triangle of module 2. These graphs have been used to display a number of useful features including the Fibonacci sequence. Additionally, the binomial networks possess various properties such as the Golden mean, Lucas number, and several other properties related to Pascal's triangle [7].

For each non-negative integer n, a binomial network B_n can be defined, whose vertices are represented by the set $\{v_j: j= 0, 1,..., 2^n\text{-}1\}$. The nodes in this graph grow exponentially at each stage, and its edges are represented by the set $E_n = \{\{v_i, v_j\}: \binom{i+j}{j} \equiv 1 (\mathrm{mod}\, 2)\}$ that can be defined for $i=0, j=0, \binom{0}{0}=1$. Therefore, every binomial network for the vertex V_0 has a self-loop; but for the rest of the vertices, it is a simple graph without self-loops. Moreover, in the binomial graph, $\binom{n}{k}$, k = 0,1,...,n-1, there is a number of vertices of degree 2^k and the degree of vertex V_0 is equal to 2^n+1. Therefore, the sum of the degrees of the vertices in the binomial network B_n can be determined from the following equation [7].

$$\begin{aligned} &\sum_{k=0}^{n-1}\binom{n}{k}2^k + (2^n+1) = 1 + \sum_{k=0}^{n}\binom{n}{k}2^k = 3^n + 1 \\ &|E_n| = \frac{1}{2}(3^n+1) \end{aligned} \tag{5}$$

The construction of the adjacency matrix for the binomial network, $A(B_n)$, is discussed in [8]. The matrix elements are obtained using the Kronecker product. To build a binomial network, a recursive relationship is used based on the previous stage, $B_{n\text{-}1}$. This recursive feature allows for the construction of networks of any order. Figure 3 illustrates the structure of a fourth-order (n=4) binomial network.

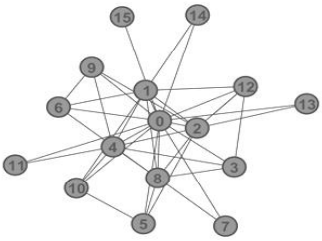

Figure 3. A binomial network of order four (n=4)

2.2.4 Fibonacci Network

Fibonacci graphs are a family of graphs derived from the Fibonacci sequence. By exploring the properties of this sequence and the relationships between its terms, researchers have identified various types of graphs with distinct characteristics and applications. In general, Fibonacci graphs can be categorized into two main types, which are described in detail below.

The first category of Fibonacci graphs is primarily used in chemistry and physics. These graphs are often derived from the molecular structure of materials, utilizing the relationship governing the Fibonacci sequence. Various polynomials can be analyzed from these graphs [9], and several studies have focused on describing their structures [10]. For instance, [10] discusses the use of independence polynomials, Fibonacci trees, and various mechanisms for implementing Fibonacci graphs to create the molecular structure of benzyl radical.

The second category of these graphs is utilized in mathematical sciences, which will also be examined in this study. In this category, the Fibonacci sequence is used on its own, resulting in the creation of Fibonacci networks. These networks are introduced below [11]. The Fibonacci series is a sequence of numbers that can be obtained from the following equation [11]:

$$F(n) := \begin{cases} 0 & if\ \ n=0 \\ 1 & if\ \ n=1 \\ F(n-1)+F(n-2) & if\ \ n>1 \end{cases} \tag{6}$$

The Fibonacci sequence is a series of numbers where each number is the sum of the two preceding ones, except for the first two numbers. The first two numbers in the series are 0 and 1, and the sequence continues as 0, 1, 1, 2, 3, 5, 8, 13, 21, and so on. These numbers are named after the 13th-century Italian mathematician Leonardo Fibonacci.

In a Fibonacci network, the number of nodes can take on any value from the Fibonacci sequence. To establish connections between the nodes, first, all the nodes in the network must be converted to their respective Fibonacci codes. To obtain the Fibonacci code of a number, one must find the first number in the Fibonacci sequence that is smaller than or equal to the given number. Then, the given number is divided by the first member of the sequence, and the remainder of the division is divided by the previous number in the sequence. This process is continued with successive divisions until the divisor becomes zero. Finally, all the quotients from the first to the last division form the first digit to the final digit in the Fibonacci code from

right to left. To better understand how to convert numbers to Fibonacci codes, an example of converting the number 7 to a Fibonacci code is provided in Figure 4.

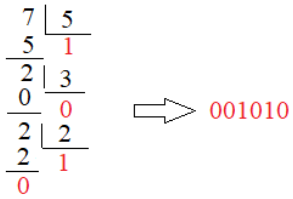

Figure 4. Converting number 7 to a Fibonacci code

Once the Fibonacci codes have been determined, the next step is to construct the adjacency matrix by establishing connections between nodes. This requires pairwise comparison of the codes, with one important consideration being that the codes must be of equal length. To ensure this, additional zeros can be appended to the end of shorter codes. After ensuring equal length of codes, comparisons are made between them. The result of each comparison determines whether the corresponding nodes should be connected or not. Specifically, if only one bit differs between two codes, then the corresponding nodes are connected, and the adjacency matrix is updated accordingly by setting the corresponding element to 1. As an example, Figure 5 shows the Fibonacci network of order 6 (i.e., n=6), which corresponds to 8 nodes.

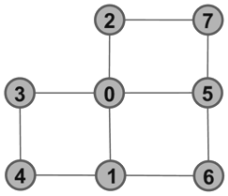

Figure 5. Fibonacci network of order six (n=6)

2.2.5 WK-Recursive Networks

WK-recursive networks are very similar to Sierpinski graphs. In fact, these networks are represented by $WK(p,n)$, which are almost equivalent to Sierpinski S_p^n. Klavžar and Milutinovic [12] have introduced such graphs named as Switching Tower of Hanoi (STH) for p pegs and n discs. These graphs are shown with symbol S_p^n where n indicates the number of vertices and p denotes the number of pegs. In this study, it is called Sierpinski graph.

Definition 3 [13]: *For* $p \in$ *and* $n \in$ $_0$ *(natural numbers greater than or equal to zero), Sierpinski graph* S_p^n *can be defined by the following equation.*

$$V(S_p^n) = p^n$$
$$E(S_p^n) = \left\{ \left\{ sij^{d-1}, sji^{d-1} \right\} | i, j \in P, i \neq j; d \in [n]; s \in p^{n-d} \right\} \quad (7)$$

Some researchers have named Sierpinski graph S_3^n as Sierpinski gasket. The structure of this network is shown in Figure 6 [13].

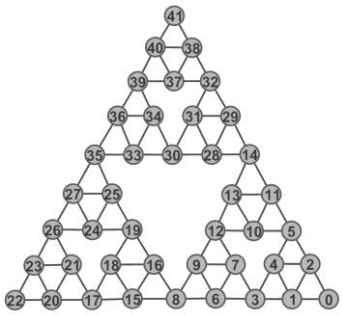

Figure 6. Sierpinski gasket network of order four (n=4)

The set of vertices in both Sierpinski and WK-recursive networks are defined by $V(WK(p,n)) = p^n = V(S_p^n)$. The only exception is that the WK(p, n) network has p additional open edges at its terminal vertices. Open edges are used for further network development. WK-recursive networks have several features that are described below [14].

WK(6,2) network, as shown in Figure 7, is an example of a network that can be used in integrated circuits. In recent years, there are many results on fault tolerance connectivities. Chen et al. [15] have extensively studied the structural properties of these networks, including diameter, connectivity, and Hamiltonian cycles, and have explored routing and broadcasting algorithms. Fang et al. [16] have also presented a simple broadcasting algorithm for such networks. These networks are also useful for designing and implementing message passing algorithms.

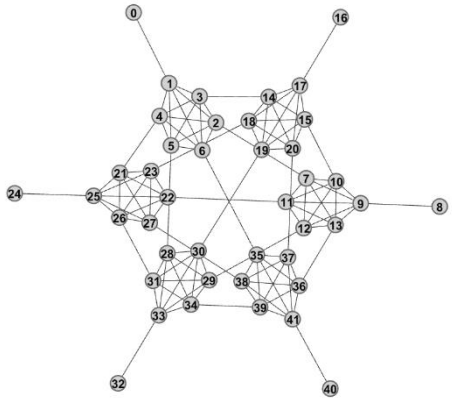

Figure 7. WK-recursive network of order 2 with six open edges

2.3 Line Graphs

In a non-empty and undirected graph G, the line graph of *G* is another graph, denoted by *L*(G) that represents the adjacencies between edges of *G*. L(*G*) is constructed such that for each edge in *G*, a vertex in L(*G*) is created and for every two edges in *G* that have a vertex in common, an edge between their corresponding vertices in *L*(G) is generated. Hence, to construct the line graph *L*(*G*), the edges in the graph *G* are mapped to the nodes and the nodes in the *G* are mapped to the edges. This simple strategy is generally used to obtain line graphs in graph theory. By transforming the graph *G* into a line graph *L*(*G*), graph clustering is improved; moreover, the line graph *L*(*G*) is more structured than the original graph *G*. The noteworthy point is that if the graph *G* has *n* vertices of degree r-regular, then the line graph *L*(*G*) will also be regular and of degree 2(*r*-1) [17]. Figure 8 shows a graph (left) and its corresponding line graph:

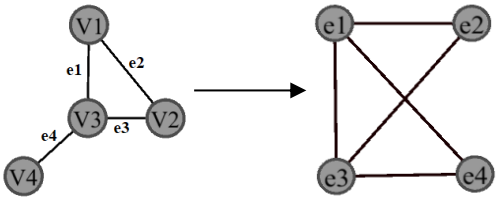

Figure 8. A graph (left) and its corresponding line graph

3. THE LITERATURE

During the past years, some of researches have been conducted with the focus of complex networks were based on criteria can better express the heterogeneity hidden in the networks.

Researchers have proposed various criteria to assess and quantify the network heterogeneity in recent years. A relatively comprehensive review of such indices, along with their respective advantages and disadvantages, has been reported in [24]. However, it is worth noting that a comprehensive review of heterogeneity measures has yet to be conducted independently. Thus, undertaking such research is an invaluable task that warrants further exploration. In this section, we introduce the five heterogeneity measures that we employ in this study. It is important to note that different heterogeneity measures can be classified into three categories; i.e., the vertex-degree-based indices (VDB) [24], the graph spectrum-based indices [23], and the information theory-based indices [24].

One of the most significant and widely used VDB heterogeneity indices is the H-index, proposed by Estrada [18]. This heterogeneity metric is defined as follows; the H-index is equal to the square of the difference of the inverse square root of the endpoint degrees of any two vertices connected by an edge in the graph. It can be written as

$$\rho'(G) = \sum_{(u,v)\in E} (d_u^{-1/2} - d_v^{-1/2})^2 \qquad (8)$$

The H-index is a standard metric, with a value of 0 (lower bound) indicating a regular graph and a value of 1 (upper bound) corresponding to star-like graphs. Its characteristics, advantages, and disadvantages are described in details in [24]. In this study, we employ the H-index to quantify the heterogeneity of recursive line graphs.

In addition to the H-index, Estrada and Estrada [19] proposed another heterogeneity index called the distance-sum; which is the second measure of the heterogeneity used in this study. The rationale behind including this index is that the total distance distribution in a network is assumed to be equivalent to the node degree distribution in the H-index. While the H-index defines the function as the square root of the node degree, the distance-sum criterion defines the function *f* as a sum of the distances of a node from all other nodes. By adjusting the desired parameter in this measure, one can obtain other indices such as the Winner (sigma) index, closeness centrality [20], and Balaban index [21].

The third heterogeneity index used in this article is the normalized degree variance, which was introduced as a heterogeneity measure by Smith and colleagues [22]. The authors claim that their proposed measure is unbiased with respect to the order and density of the underlying graphs. However, according to [24], the H-index proposed by Estrada [18] has two main drawbacks. First, for quasi-complete graphs that contain isolated nodes, the H-index can lead to division by zero, resulting in an error. Second, it is sensitive to the order and density of the networks, making it a skewed index.

The heterogeneity measures discussed above belong to the vertex-degree-based (VDB) category and are incompatible with each other. In this study, a fourth heterogeneity index, based on the network spectrum, was employed. Safaei et al. [23] introduced a normalized measure to assess the heterogeneity properties of graphs complex networks; which is dependent on the spectral theory of graphs. The authors demonstrated that their proposed measure has a lower bound of 0 and an upper bound of 1 for regular and star-like graphs, respectively. Through simulation experiments, the proposed energy index was shown to accurately explore the structural differences of networks while maintaining low computational complexity.

In [24], Emadi et al. introduced a novel and efficient approach based on discrete and generalized graph entropies-energies. They proposed several heterogeneity measures to quantify the structural heterogeneity properties of underlying graphs. The proposed measures can also be used to classify and compare graphs with different structures. We employed such measures as the fifth index of network heterogeneity in the current study. The authors demonstrated how the heterogeneity measures such as Shannon entropy, von Neumann entropy (quantum entropy), and the generalized graph entropies-energies can be utilized to evaluate the degree of heterogeneity in graphs and complex networks.

4. EXPERIMENTAL RESULTS

In this section, we begin by analyzing the frequency distribution of eigenvalues in recursive line graphs. A key question that arises is whether the line graph of Erdős-Rényi (ER) networks also follows a binomial distribution, given that the degree distribution of the original ER graph is binomial. To address this question, researchers have studied a specific distribution of line graphs. In a previous study [17], the authors demonstrated that the degree distribution of the original graph and its corresponding line graph are highly similar and cannot be distinguished from each other.

The researchers found that the distribution of eigenvalues in ER line graphs is noticeably different from that of their original graphs, despite the degree distributions of the two being very similar and indistinguishable from each other. Specifically, while the eigenvalues of ER graphs follow a semicircular distribution, the eigenvalues of their line graphs do not. This suggests that the line graph of a network is distinct from its main network in terms of eigenvalue distribution. Figure 9 illustrates the distribution of eigenvalues for both recursive graphs and their corresponding recursive line graphs.

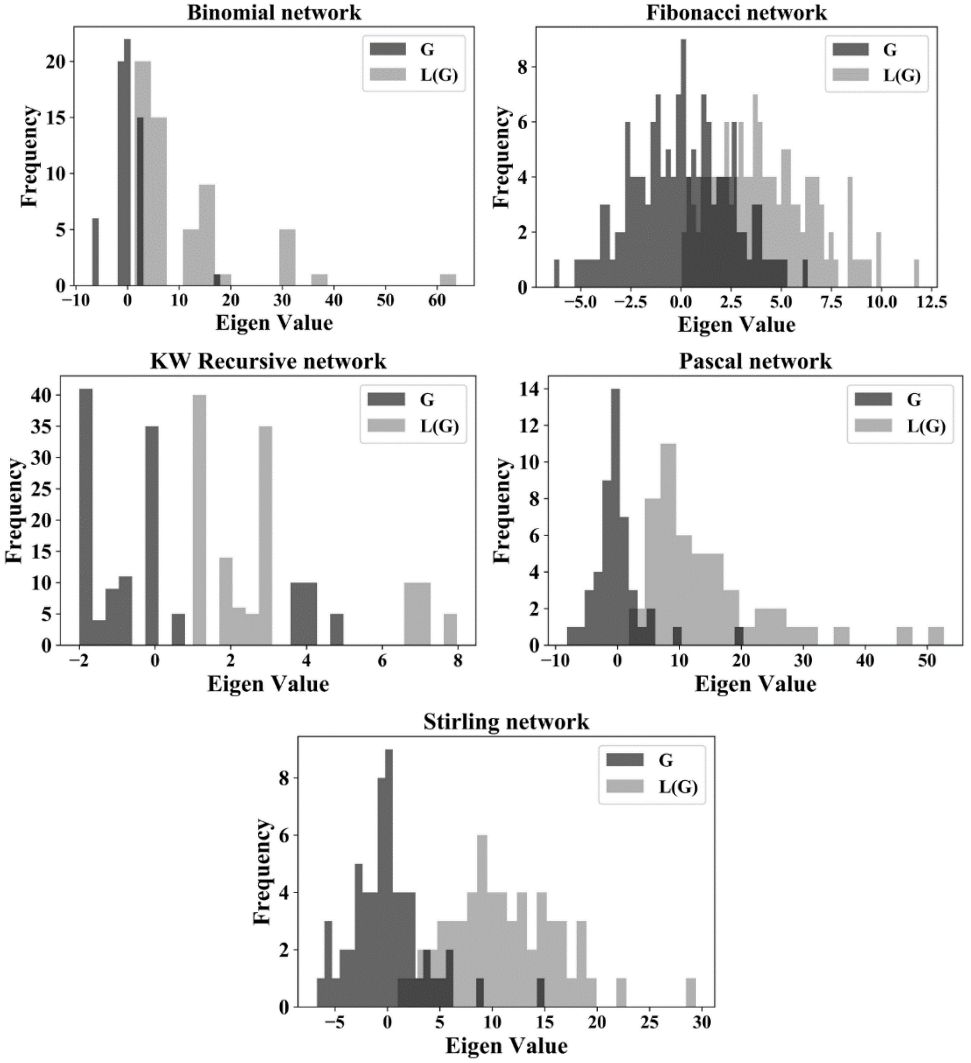

Figure 9. Distribution of the eigenvalues in recursive networks and the corresponding recursive line graphs

In the following analysis, we aim to determine whether five commonly used heterogeneity measures can effectively distinguish the structural complexity of line graphs in recursive networks. To accomplish this, we have plotted numerical values obtained

by the five heterogeneity measures as a function of the size of the corresponding line graph in Figure 10. The average values for each of the five heterogeneity measures were calculated and presented in the form of a linear diagram based on empirical samples obtained from simulation experiments.

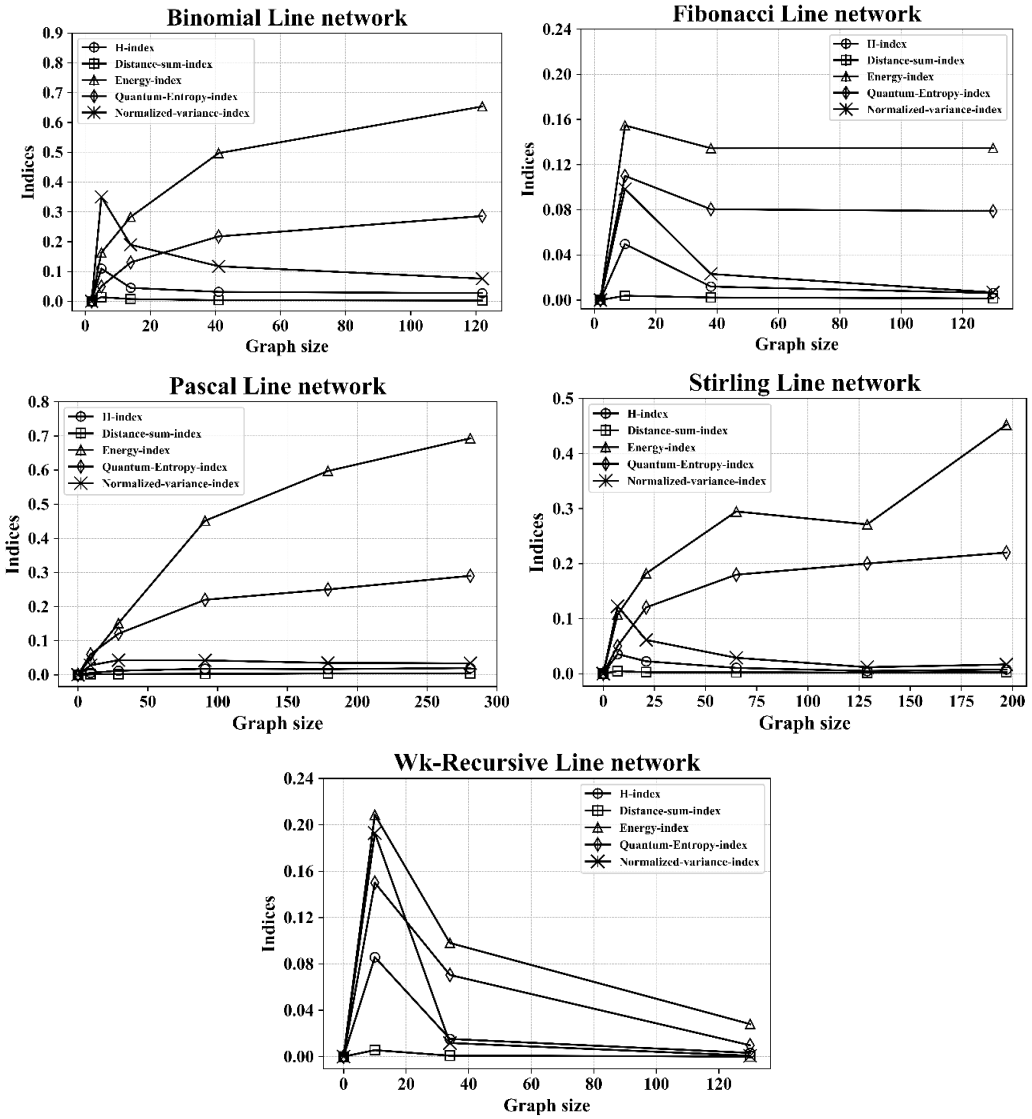

Figure 10. Comparison of the irregularity measures for binomial, Fibonacci, Pascal, Stirling and WK-recursive networks

Based on the information presented in Figure 10, it can be observed which of the heterogeneity measures can effectively differentiate between the corresponding recursive line graphs. As the size of the recursive line networks increases, the level of heterogeneity tends to decrease. However, it should be noted that for larger recursive line networks, the amount of heterogeneity would tend to be a relatively constant quantity. The important finding from the diagrams is that even in recursive networks, the significant heterogeneity is not very distinct. Therefore, heterogeneity indices can only report a small fraction of the heterogeneity of star-like graphs. Another crucial point is that for a certain size of recursive line graphs, the heterogeneity values do not differ significantly from each other. It is essential to mention that the heterogeneity indices resulting from the parameter setting are often much lower than what occurs due to the difference in the models' structural complexity. Furthermore, the diagrams demonstrate that the heterogeneity index derived from graph energy has been more successful than the other heterogeneity measures in distinguishing between different recursive line graphs. It is evident that an important application of any suitable heterogeneity measure is whether the related metric can provide a proper classification of the recursive line graphs.

In addition to heterogeneity measures, the synchronization parameter, *Q*, can also be used as a parameter for network analysis. *Q* is calculated by taking the ratio of the second smallest non-zero eigenvalue of the Laplacian matrix m_2 to the largest eigenvalue of the same matrix μ_n. In this section, we aim to investigate the potential use of this parameter alongside other heterogeneity measures through a qualitative experiment.

In order to accomplish this, we have referred to the report by Estrada [25] on the paradox of heterogeneity. This report explains how increasing the heterogeneity of a network can actually result in a decrease in its synchronization degree and average geodesic distance. The issue of synchronization is critical in network dynamics and has significant implications for machine learning and wireless sensor networks. To assess the level of heterogeneity in the recursive line networks, we have calculated the synchronization parameter (*Q*) along with other heterogeneity indices, as shown in Table 1. Additionally, we have calculated

Pearson's correlation coefficient to determine the correlation between the natural logarithm of each heterogeneity measure and the natural logarithm of *Q*. The slope and the intercept of the fitted regression line using the natural logarithm of *Q* and the natural logarithm of the desired heterogeneity measure are expressed in the following equation.

$$Q = \mu_2 / \mu_n \tag{9}$$

Table 1. The heterogeneity indices of recursive line networks

Network		*N*	M	Q	Skewness	Kurtosis	H index	Distance sum	Energy index	Normalized variance index	quantum Entropy index
Binomial		365	7448	0.1069	0.8284	3.0092	0.0378	0.0024	0.7829	0.0499	0.0263
Fibonacci		420	2131	0.0368	0.6847	3.0092	0.0059	0.0009	0.1332	0.0021	0.0038
WK-Recursive		315	1250	0.0115	-7.7470	61.0161	0.0003	0.00006	0.0141	0.0002	0.0015
Pascal		409	8571	0.0820	0.5644	2.7610	0.0285	0.0056	0.7291	0.0277	0.0181
Stirling		389	5157	0.0339	-0.8419	3.7014	0.0007	0.0021	0.4724	0.0077	0.0044
Statistics	Pearson correlation				0.8476	0.8553	-0.7694	-0.8880	-0.9585	-0.9600	-0.9786
	Slope						-0.2176	-0.8182	-0.2951	-0.2480	-0.1880
	Intercept						-7.8865	-8.9207	-3.6522	-6.7722	-5.9793

In addition, Estrada [25] also highlights the significant relationship between the synchronization parameter, *Q*, and the clustering coefficient in terms of the network's robustness, making it a valuable index for analyzing the resilience of graphs and complex networks.

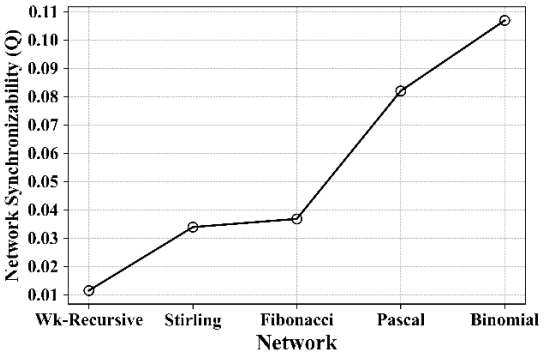

Figure 11. Variations of the synchronization parameter (*Q*) in recursive line networks

Figure 11 depicts the values of the synchronization parameter *Q* for the recursive line networks. This visualization provides a better understanding of the changes in the *Q* parameter across different networks. The WK-recursive line network exhibits the lowest value of the synchronization parameter *Q*, while the binomial line network has the highest value, indicating convergence in this network.

5. CONCLUSIONS AND FUTURE WORK

Communication networks play a crucial role in facilitating various human activities, including specialized tasks and personal well-being. However, complex and social networks, which are integral to our daily lives, are often overlooked in terms of their ability to serve these purposes and are vulnerable to a range of challenges. In recent years, several indicators have been introduced to estimate the level of heterogeneity in networks, with many of these indicators being based on the degree of nodes in the graph. In this article, we utilized a set of key measures to assess the heterogeneity present in line networks that correspond to various types of recursive networks, including binomial, Pascal, WK, Fibonacci, Stirling, and WK-recursive networks. Through a range of simulation experiments, we demonstrated the effectiveness of these measures in quantifying and evaluating the level of heterogeneity in the structure of recursive line networks. It is worth noting that heterogeneity in the structure of recursive line networks can trigger structural changes in the matrices derived from such graphs. Further research can build upon the work presented in this paper by developing a novel and efficient heterogeneity measure that explores the structural complexities of recursive line graphs from a heterogeneity perspective.

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